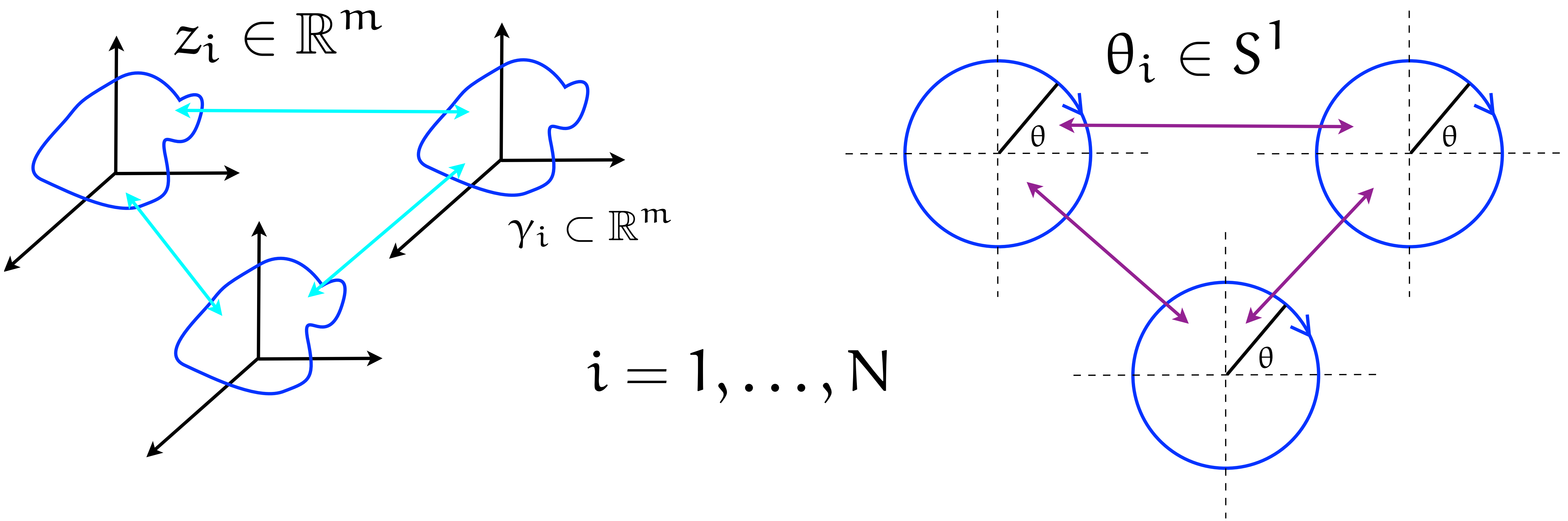
Weakly Coupled Oscillator Theory



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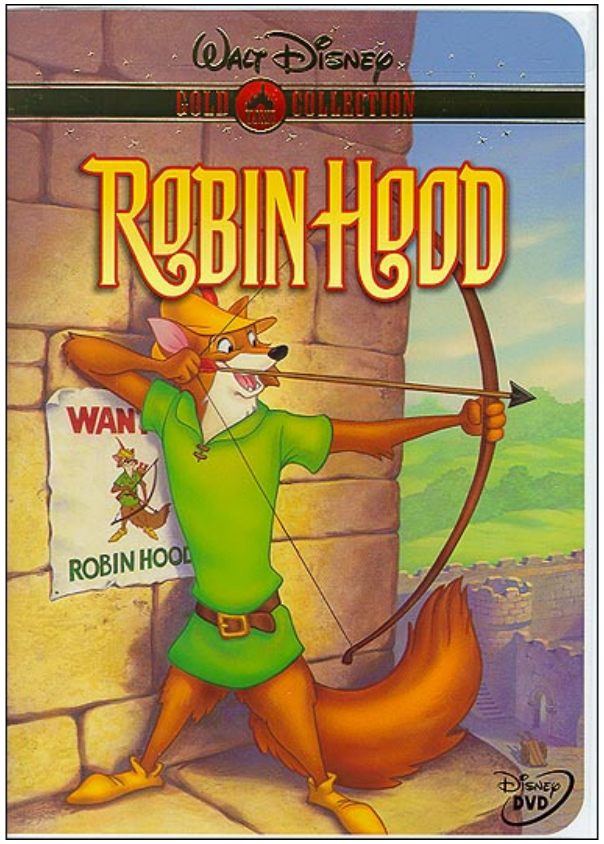
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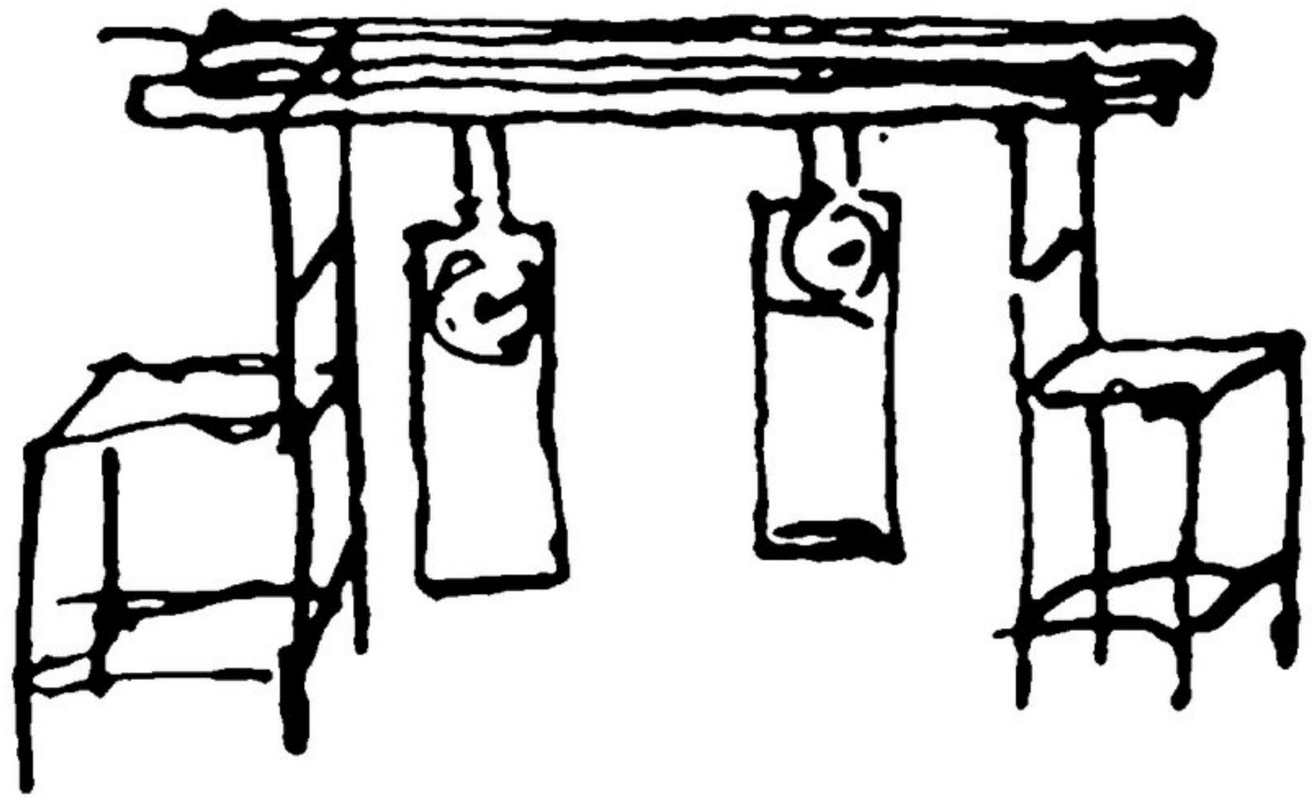
An oscillator network (*an odd kind of sympathy*)



It is quite worth noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible. Christiaan Huygens 17th Century.

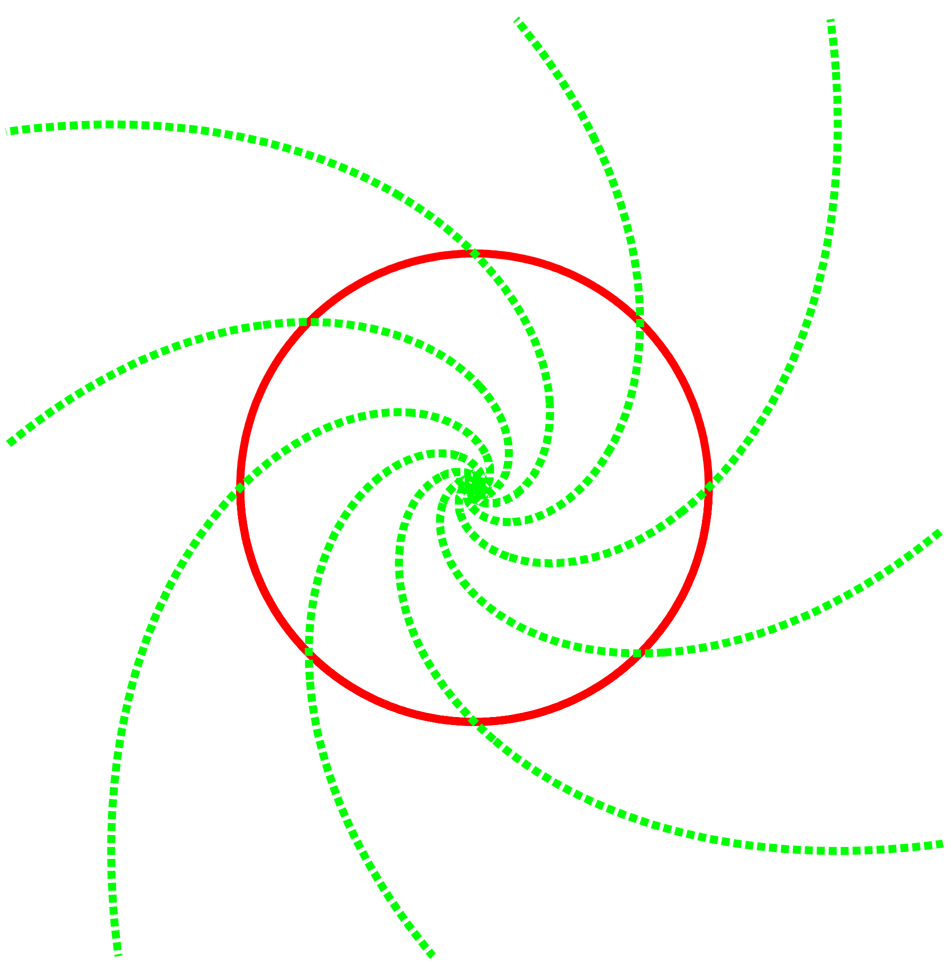
For videos of synchronising metronomes see https://www.youtube.com/watch?v=W1TMZASCR-I

A generic oscillator: Stuart-Landau model

$$\dot{z} = (1 + i\eta)z - (1 + i\alpha)z|z|^2$$

Rewriting in polar coordinates using $z = Re^{i\theta}$ gives

$$\dot{R} = R(1 - R^2), \qquad \dot{\theta} = \eta - \alpha R^2,$$



It is common practice to characterise an oscillator in terms of its phase response to a perturbation.

This gives rise to the notion of a so-called phase response curve (**PRC**).

Isochrons and phase response

Dynamical system: $\dot{x} = f(x), \qquad x \in \mathbb{R}^n$

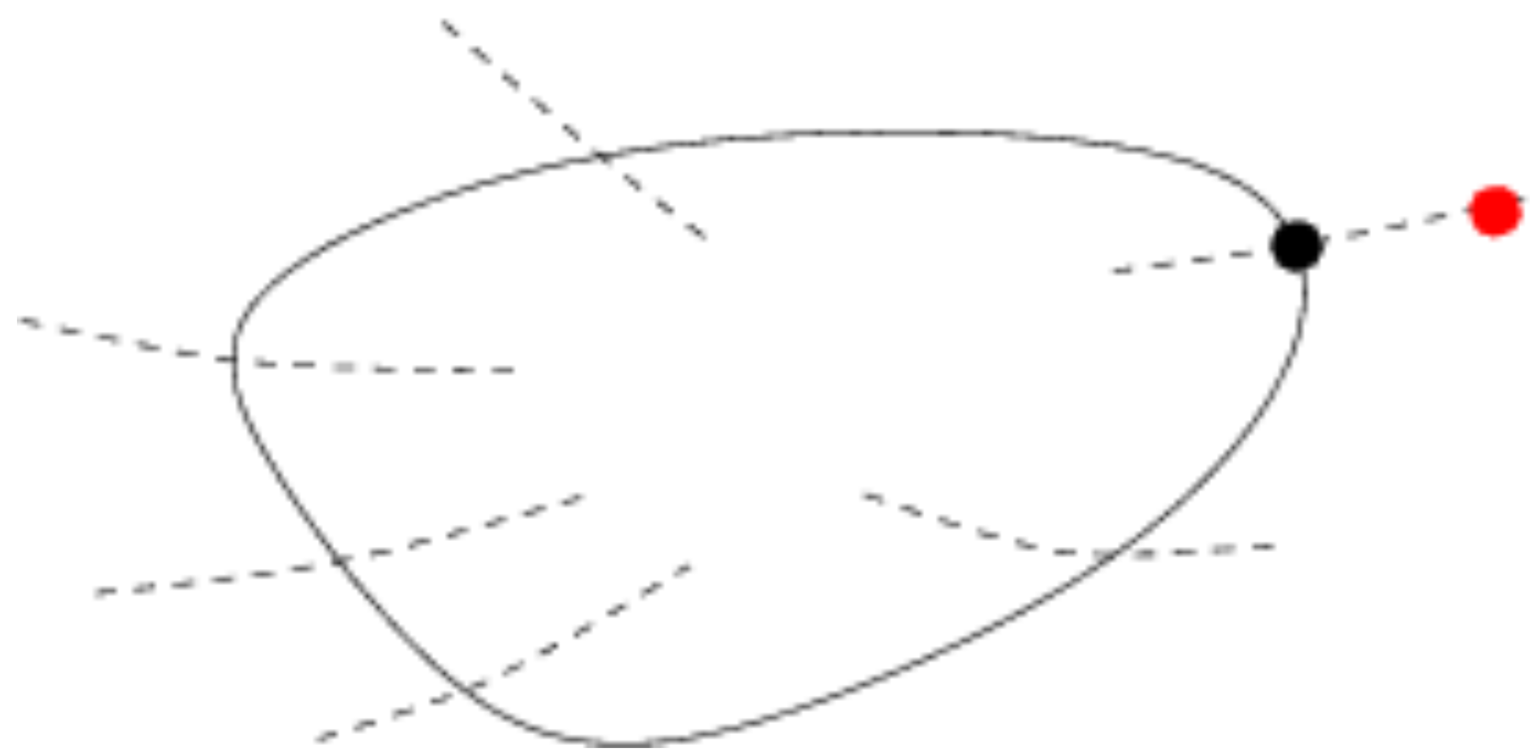
Stable limit cycle: $\Gamma = \{x_0(t) \in \mathbb{R}^n; t \in \mathbb{R}\}$

$$x_0(t + \Delta) = x_0(t) \qquad \omega = \frac{2\pi}{\Delta}$$

Assign a phase coordinate ϕ to each point $x \in \Gamma$ according to

$$\frac{d}{dt}\phi(x) = \omega$$

An isochron extends the notion of phase to *off-cycle*.



Isochrons for separable plane-polar models

$$\dot{r} = F(r), \qquad \dot{\theta} = \omega(r)$$

Suppose that there is a stable limit cycle at $r = r_0$ with angular frequency $\omega = \omega(r_0)$

By polar symmetry, the equation for an isochron is

$$\phi = \text{constant} = \theta - f(r), \qquad f(r_0) = 0$$

Differentiate with respect to t

$$\dot{\phi} \equiv \omega(r_0) = \dot{\theta} - \frac{df}{dr}\dot{r}$$

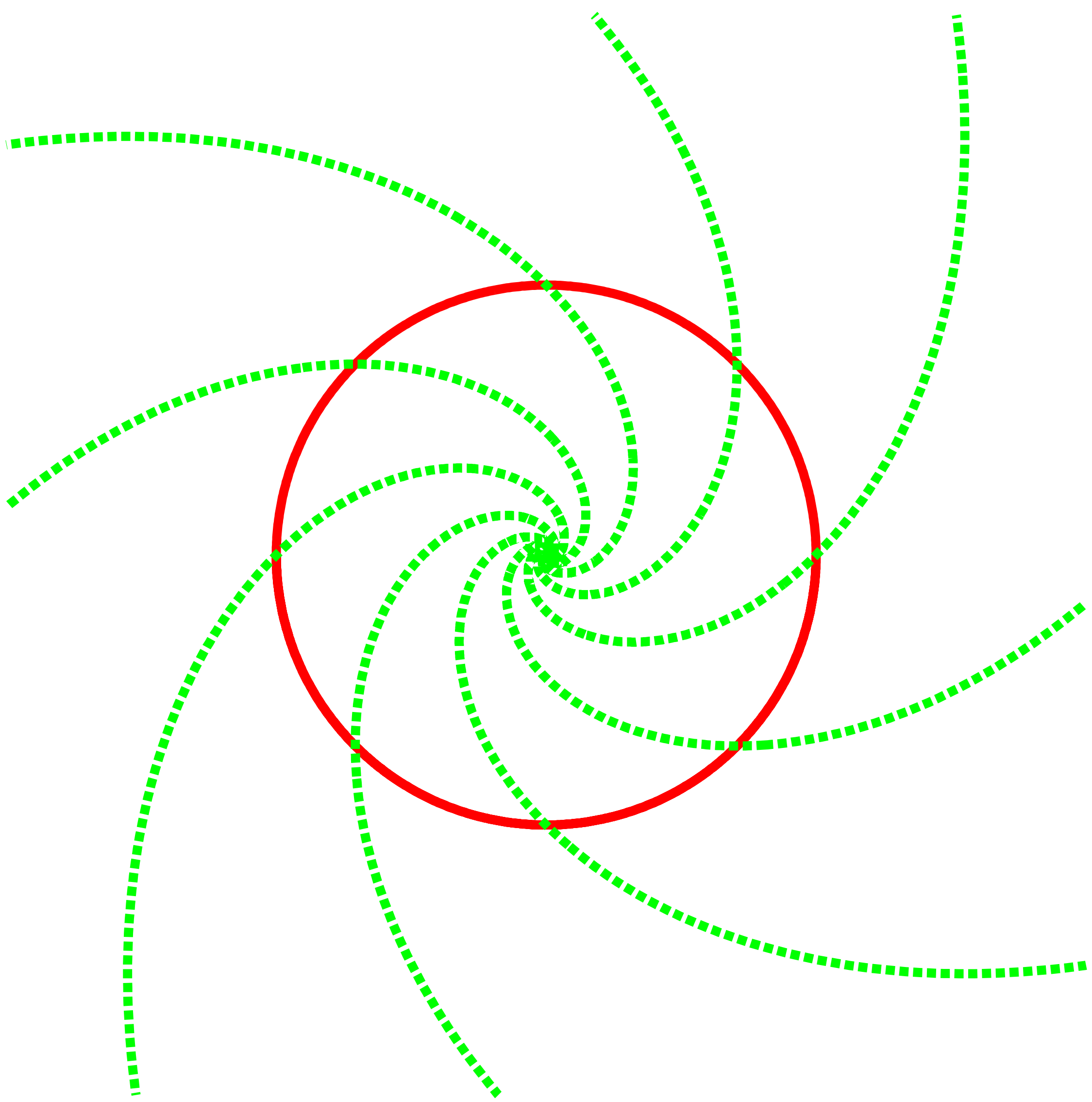
Hence

$$\frac{df}{dr} = \frac{\omega(r) - \omega(r_0)}{F(r)}$$

Isochrons for Stuart-Landau model

$$F(R) = R(1 - R^2), \qquad \omega(R) = \eta - \alpha R^2$$

$$\frac{df}{dR} = \frac{\omega(R) - \omega(R_0)}{F(R)} = \frac{\eta - \alpha R^2 - (\eta - \alpha)}{R(1 - R^2)} = \alpha \frac{1 - R^2}{R(1 - R^2)} = \frac{\alpha}{R}$$



$$\phi = \theta - \alpha \log R$$

Isochrons are *spirals*.

Phase response

iPRC $Q = \nabla_x \phi$

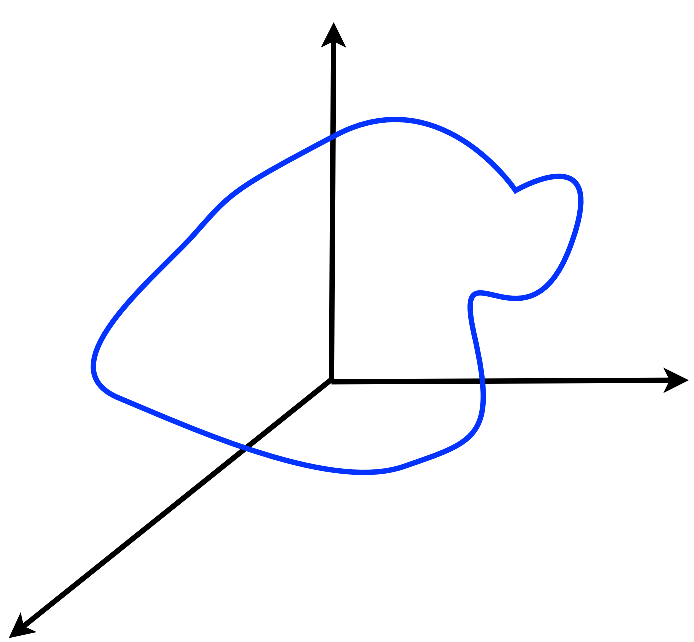
$$\phi(x + \delta x) = \phi(x) + \langle \nabla_x \phi(x), \delta x \rangle + \dots$$

In general iPRC or *adjoint* satisfies

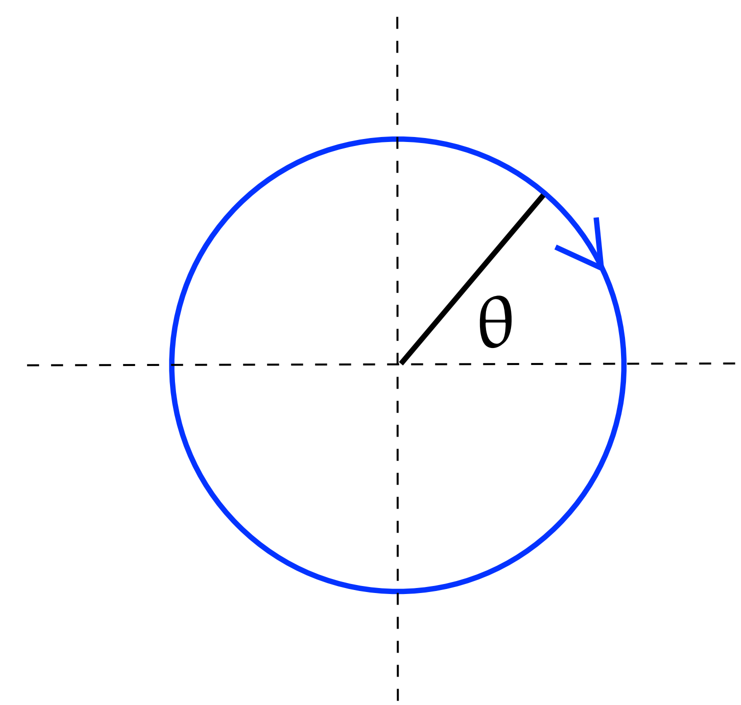
$$\frac{d}{dt} Q = -Df^{\top}(x_0(t))Q, \qquad \langle Q(x_0), f(x_0(0)) \rangle = \omega$$

Ensures

$$\frac{d}{dt}\phi(x_0) = \langle \nabla_{x_0}\phi(x_0), \frac{d}{dt}x_0 \rangle = \omega$$



$$\frac{d}{dt}\theta = \omega$$



Phase dynamics

Effect of a small external periodic force on the self sustained oscillations:

$$T \neq \Delta$$

$$\dot{x} = f(x) + \epsilon p(x,t), \qquad p(x,t) = p(x, t+T)$$

$$\dot{\phi} = \langle \nabla_x \phi, \dot{x} \rangle = \langle \nabla_x \phi, f(x) + \epsilon p(x,t) \rangle = \omega + \epsilon \langle \nabla_x \phi, p(x,t) \rangle$$

To a first approximation evaluate the rhs on the limit-cycle to get the phase dynamics:

$$\dot{\phi} = \omega + \epsilon I(\phi, t), \qquad I(\phi, t) = \langle Q(x_0), p(x_0, t) \rangle$$

Averaging

Phase dynamics still hard to analyse :(

Rotating frame $\psi = \phi - 2\pi t/T$

Detuning

$$\dot{\psi} = -\Delta\omega + \epsilon I(\psi + 2\pi t/T, t), \qquad \Delta\omega = \frac{2\pi}{T} - \omega$$

For ϵ and $\Delta\omega$ small, ψ evolves slowly and we may *average* over one period

$$\dot{\psi} \simeq -\Delta\omega + \epsilon H(\psi)$$

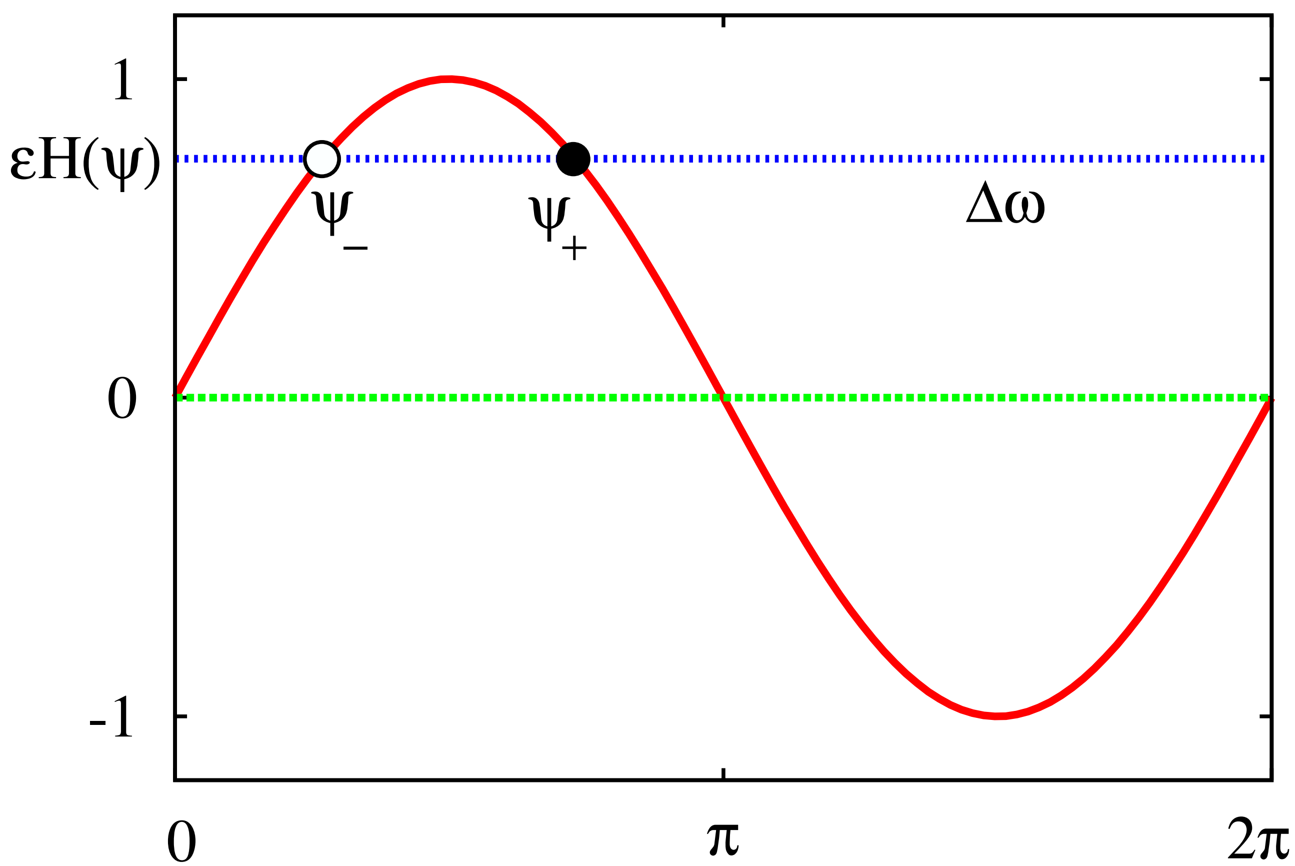
$$H(\psi) = \frac{1}{T}\int_0^T \langle Q(x_0(\psi + 2\pi s/T)), p(x_0(\psi + 2\pi s/T), s)\rangle \, ds$$

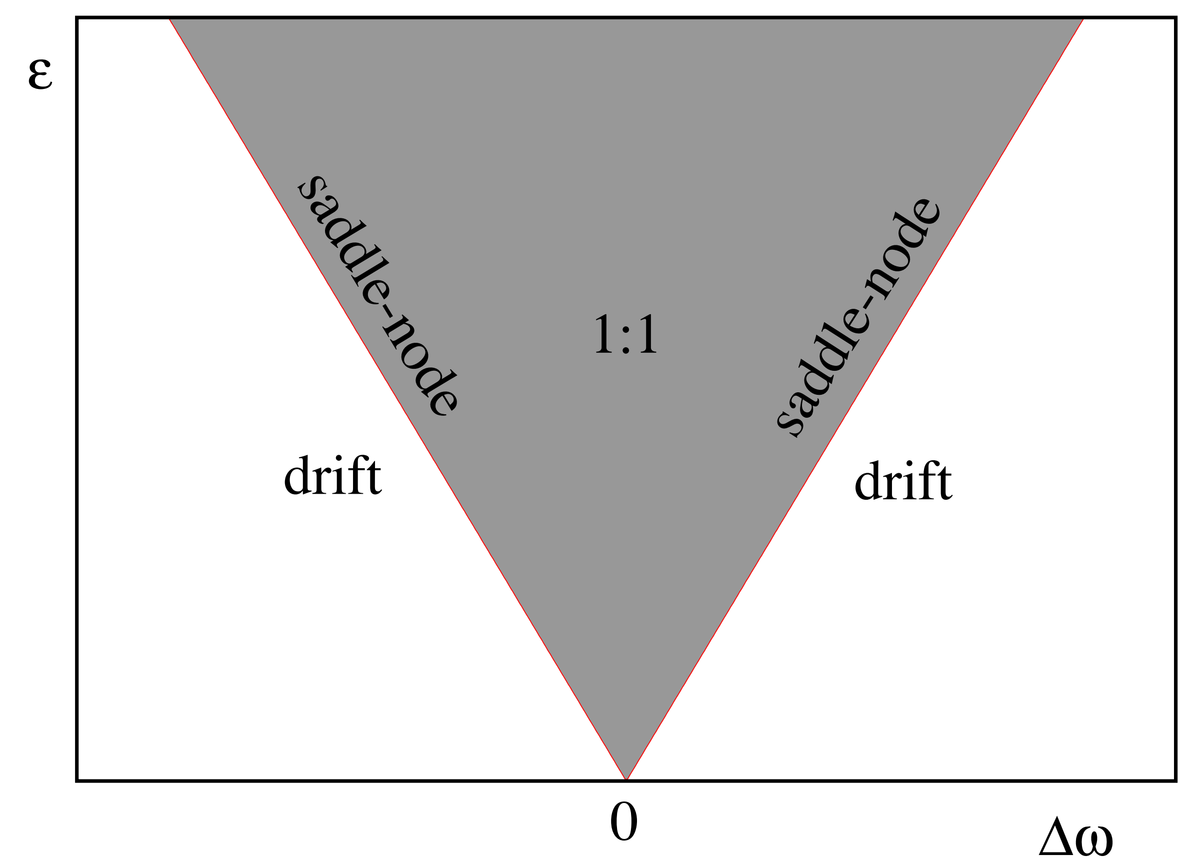
Phase Interaction Function

Adler equation - classic example

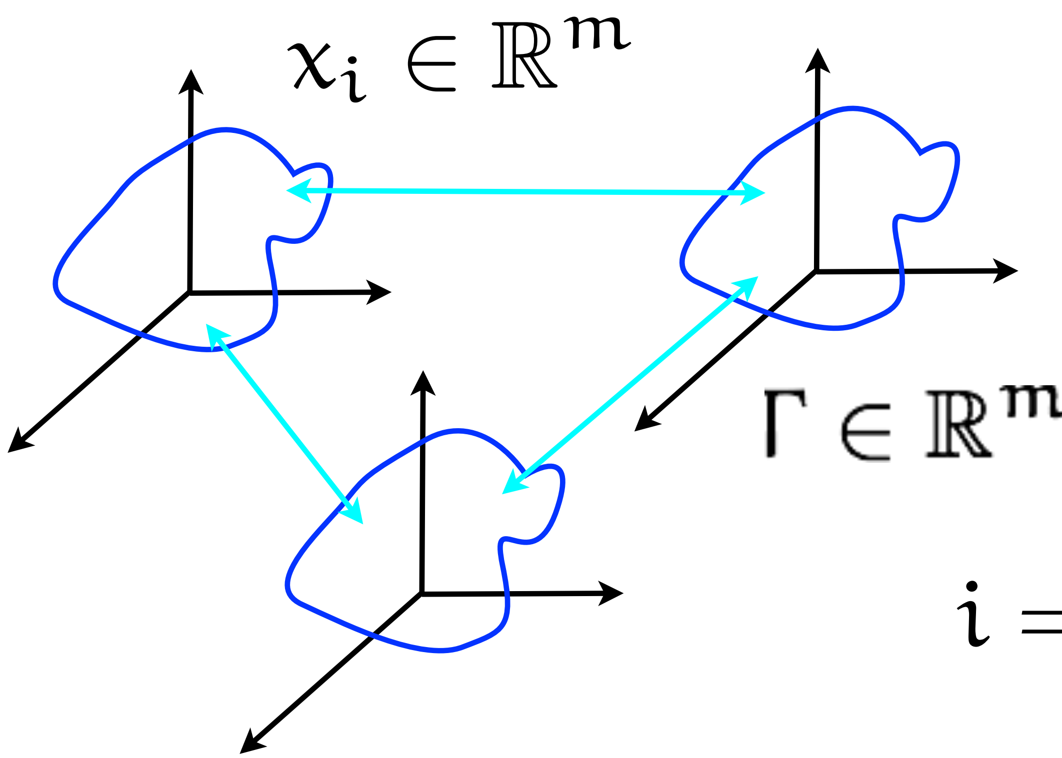
$$H(\psi) = \sin\psi$$

Synchronisation condition $\left|\frac{\Delta\omega}{\epsilon}\right| < 1$

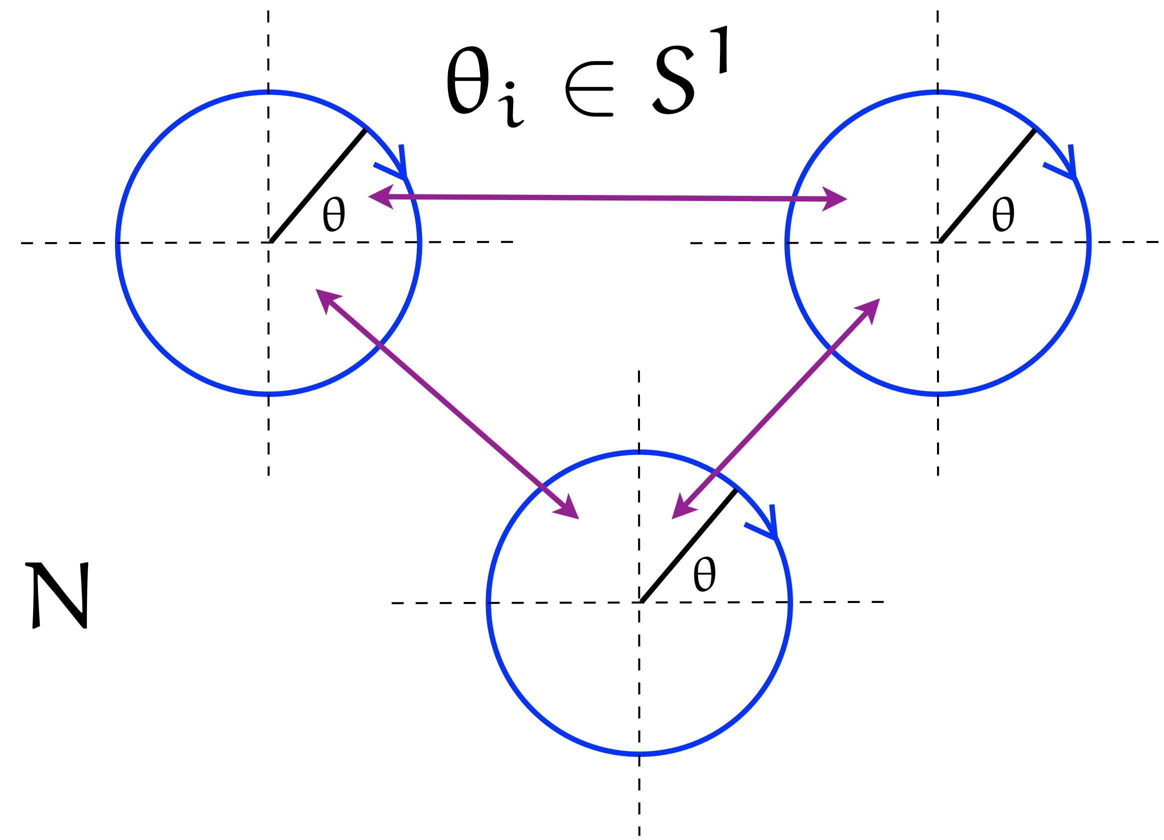




Weakly Coupled Network



$i = 1, \ldots, N$



$$\dot{x}_i = f(x_i) + \epsilon \sum_{j=1}^{N} w_{ij} G(x_j)$$

Uncoupled system has an exponentially stable limit cycle Γ

Direct product of hyperbolic limit cycles is a normally hyperbolic invariant manifold

$$\dot{\theta}_i = \omega + \epsilon \sum_{j=1}^{N} w_{ij} \langle Q(\theta_i), G(\theta_j) \rangle$$

Drive

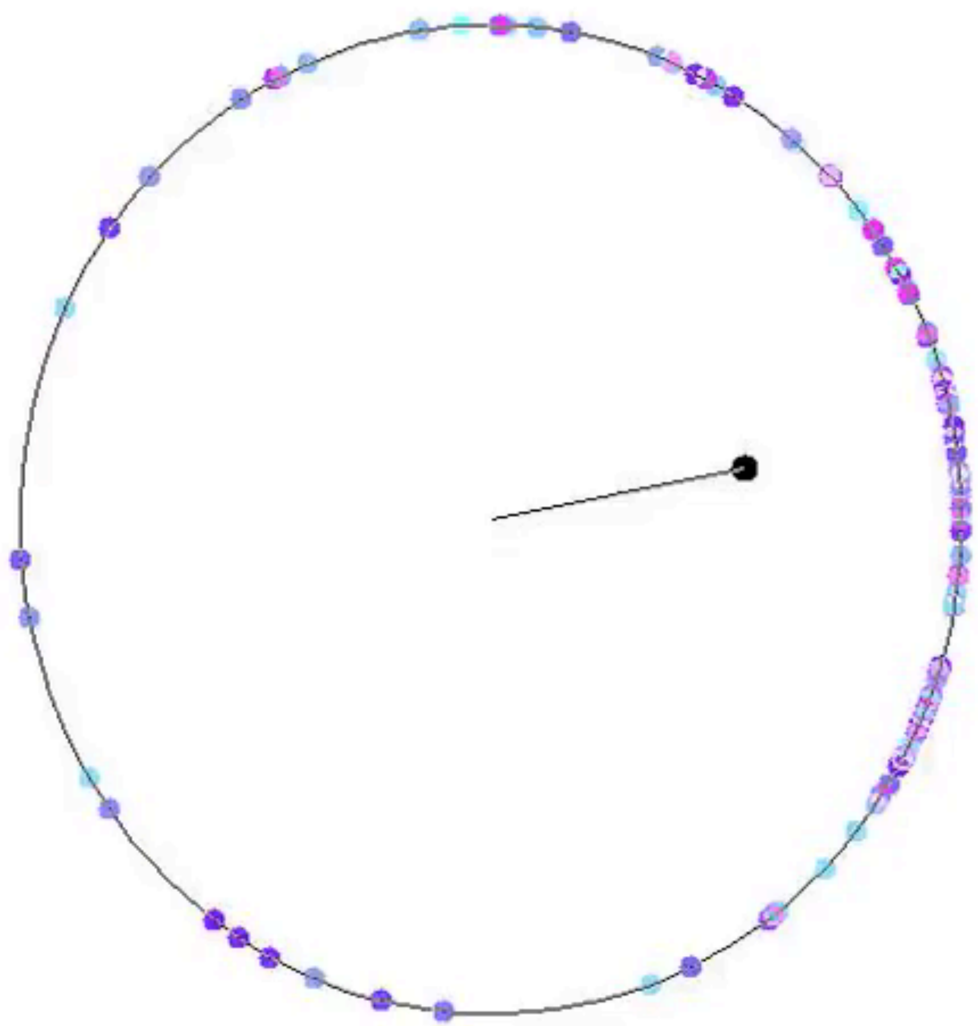
iPRC

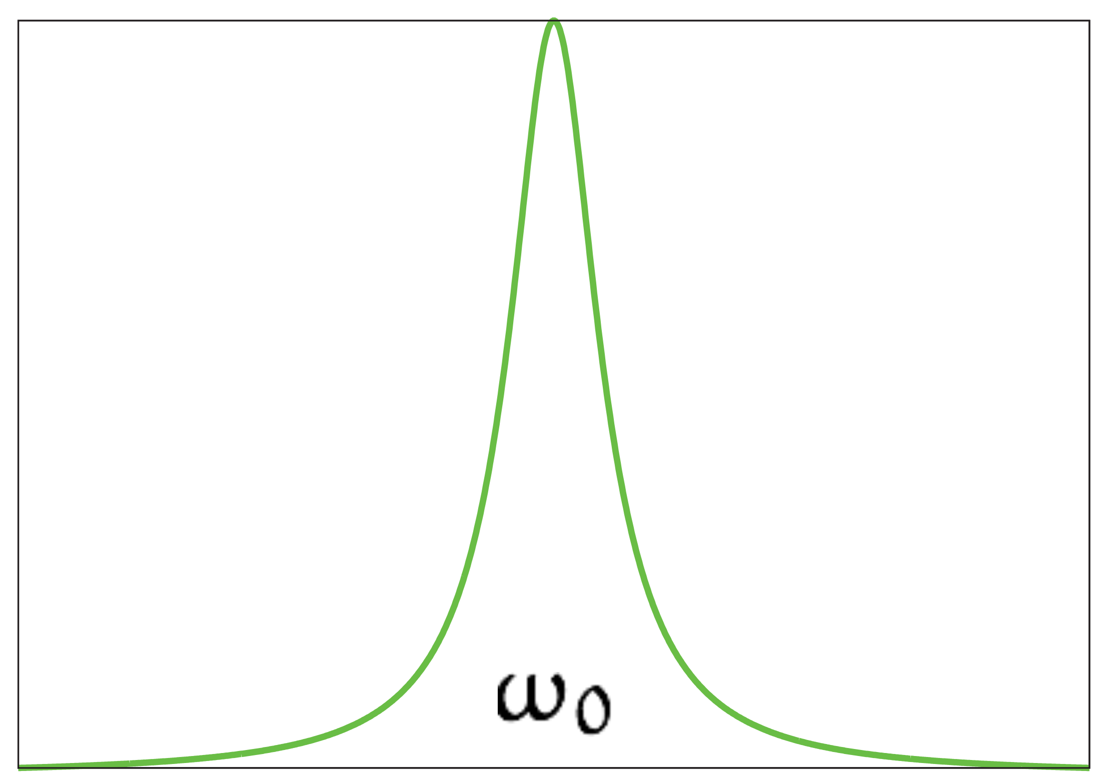
Averaging: *phase-difference network*

$$\dot{\theta}_i = \omega + \epsilon \sum_{j=1}^{N} w_{ij} H(\theta_j - \theta_i) \rangle$$

Kuramoto network: $w_{ij} = \frac{1}{N}$

$$H(\theta) = \sin(\theta)$$





General phase-locked states

Phase-locked state: $\theta_i(t) = \phi_i + \Omega t$

constant → ϕ_i emergent frequency → Ω

$$\Omega = \omega + \epsilon \sum_{j=1}^{N} w_{ij} H(\phi_j - \phi_i), \qquad i = 1, \ldots, N$$

After choosing some reference oscillator, these N equations determine the collective frequency Ω and $N - 1$ relative phases.

Linear stability: $\theta_i(t) = \phi_i + \Omega t + \widetilde{\theta}_i(t)$

$$\frac{d}{dt}\widetilde{\theta}_i = \sum_j \widehat{\mathcal{H}}_{ij}(\Phi)\widetilde{\theta}_j$$

$$\Phi = (\phi_1, \ldots, \phi_N)$$

Jacobian $\widehat{\mathcal{H}}(\Phi)$

$$\widehat{\mathcal{H}}_{ij}(\Phi) = \epsilon \left[w_{ij} H'(\phi_j - \phi_i) - \delta_{ij} \sum_k w_{ik} H'(\phi_k - \phi_i) \right]$$

Synchrony $\theta_1 = \theta_2 = \ldots = \theta_{N-1} = \theta_N, \quad \dot{\theta}_i = \Omega$

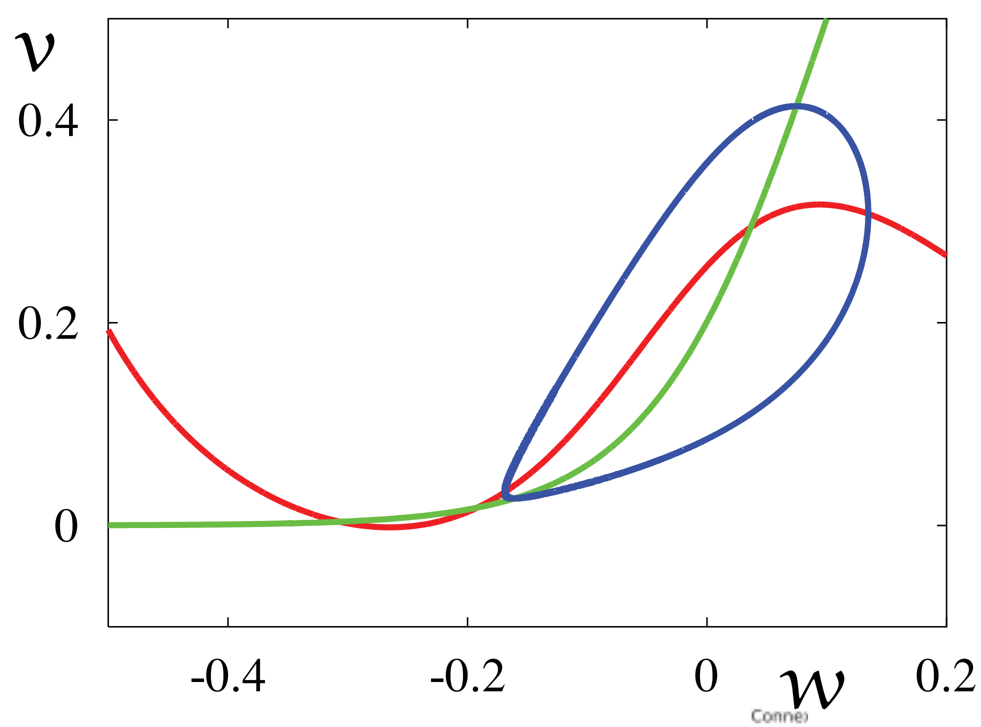
$$\Omega = \omega + \epsilon H(0) \sum_{j=1}^{N} w_{ij}, \qquad \forall i$$

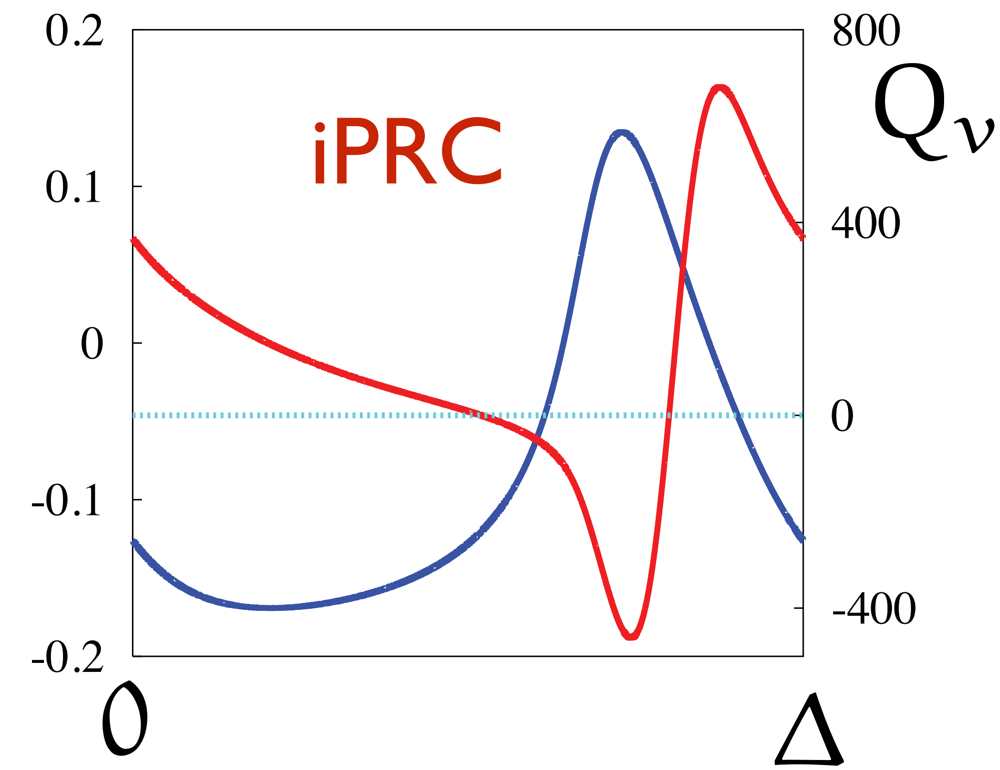
Solutions exist for *diffusive coupling* $H(0) = 0$ *or* $\sum_j w_{ij} = \text{const}$

$$\widehat{\mathcal{H}} = -\epsilon \mathcal{G} H'(0), \qquad \mathcal{G}_{ij} = -w_{ij} + \delta_{ij} \sum_k w_{ik}$$

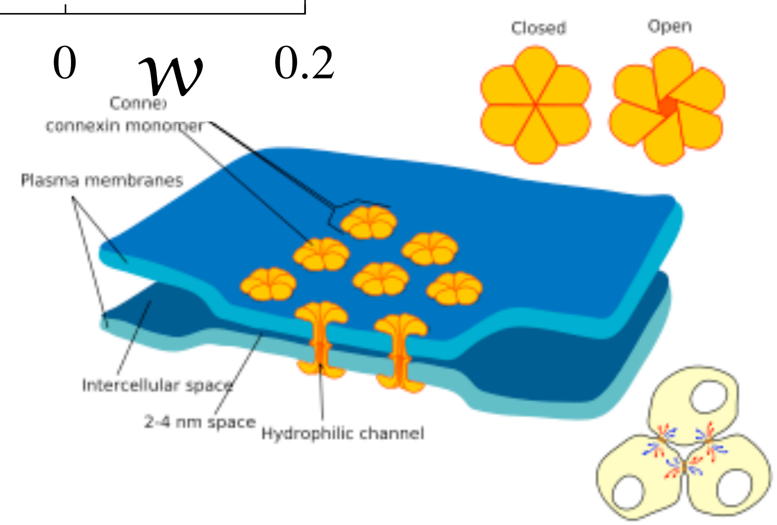
$\mathcal{G} = \text{Graph Laplacian}$

An example - Morris-Lecar network





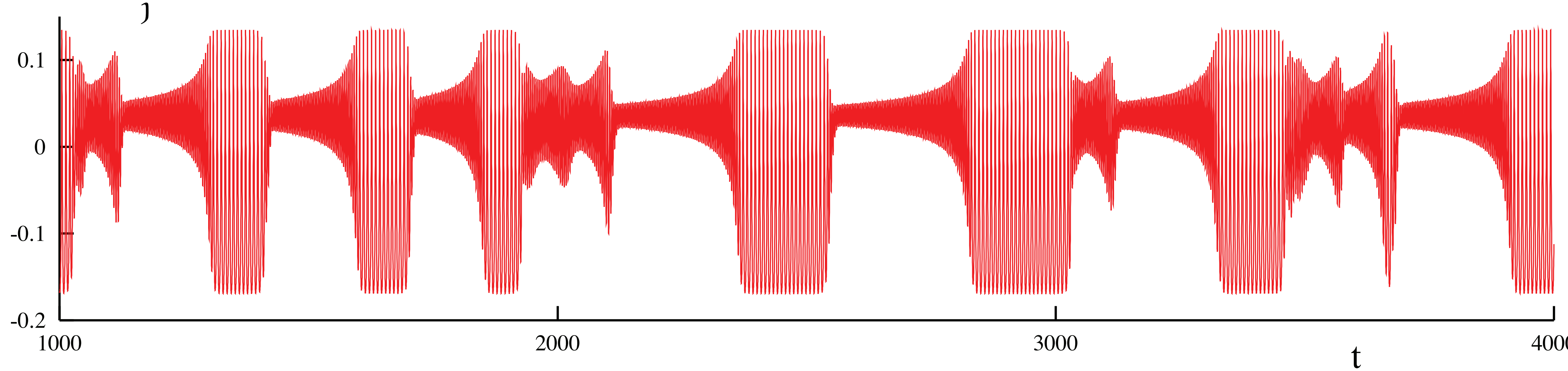
gap junction coupling



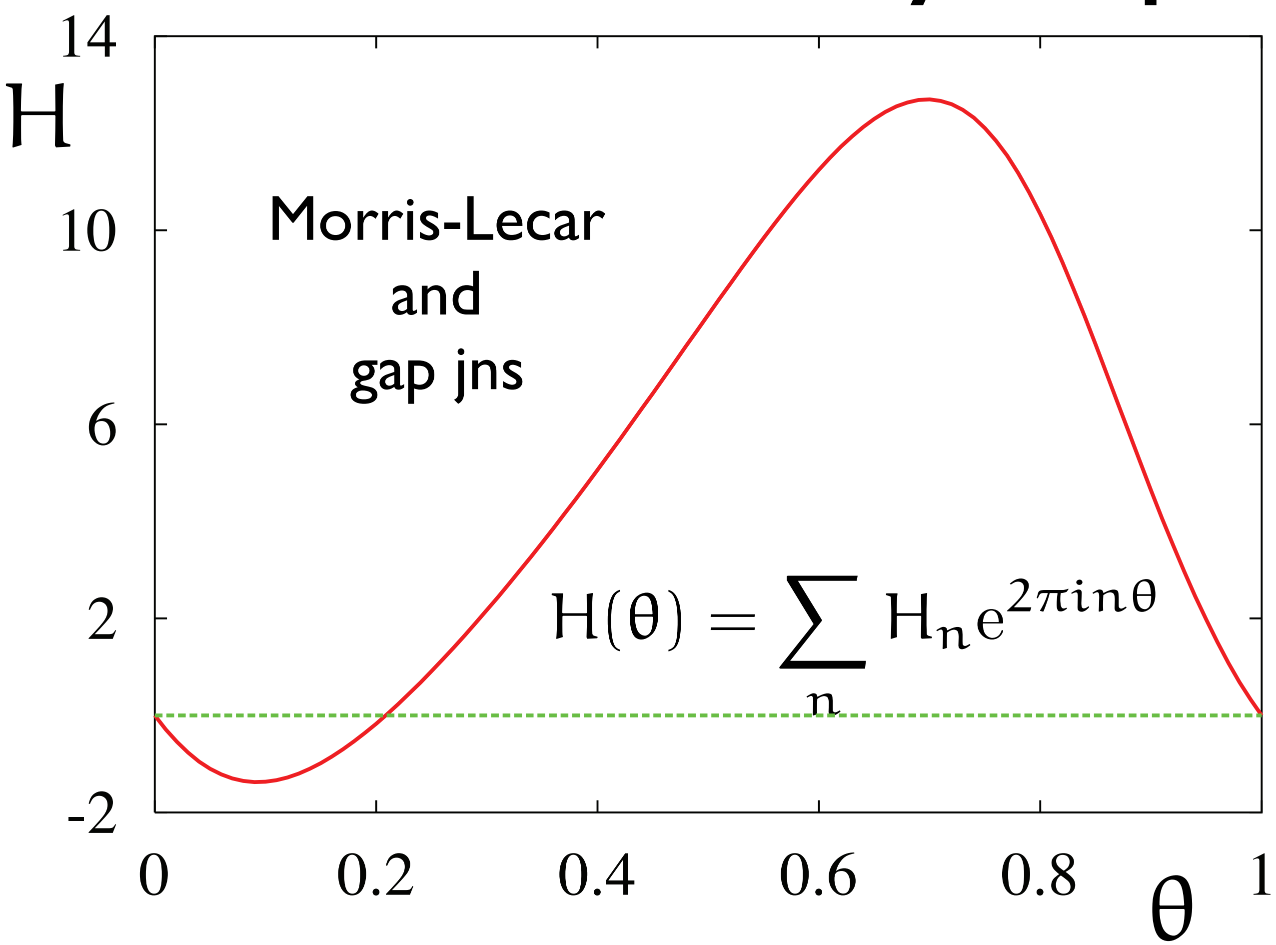
$$\frac{1}{N}\sum_{j=1}^{N}(v_j - v_i)$$

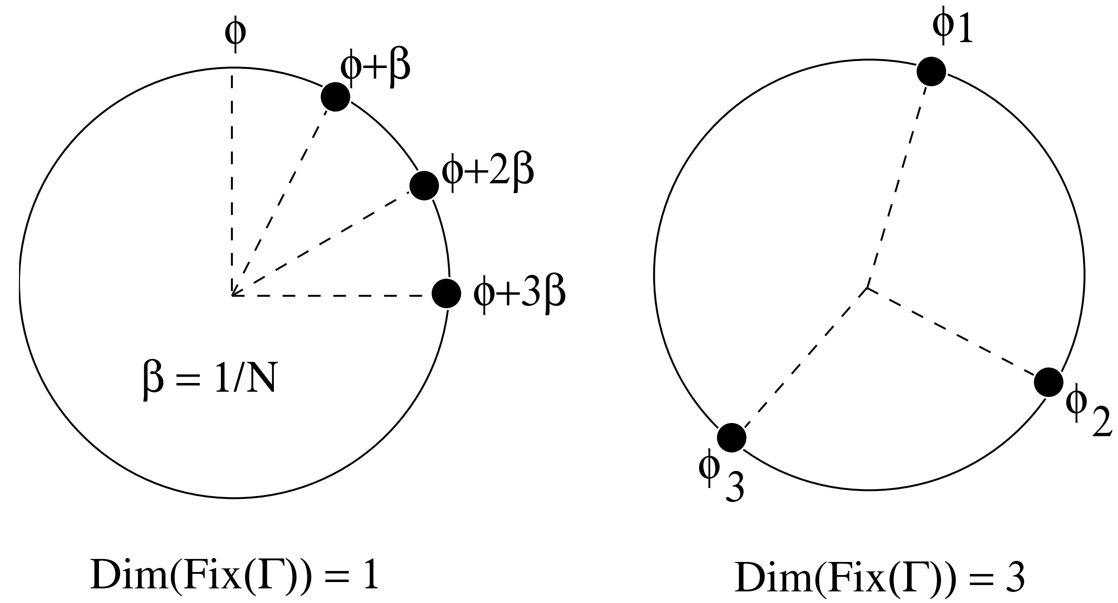
$$E = \frac{1}{N}\sum_j v_j$$

Morris-Lecar



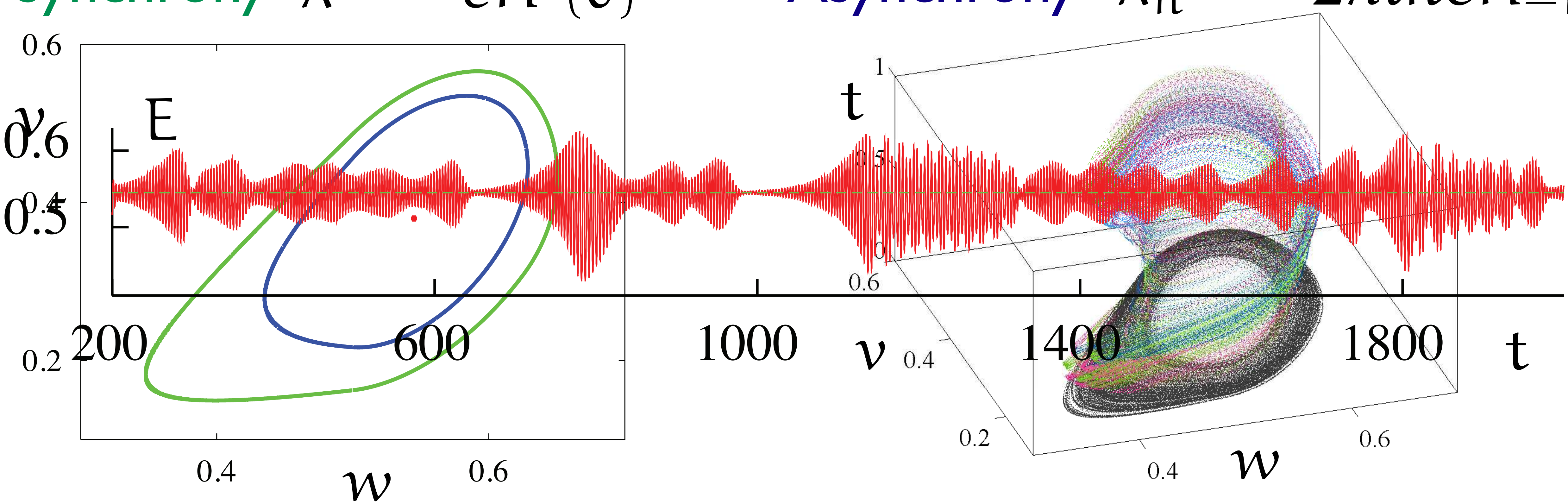
Stability of phase-locked states





Bifurcations from maximally symmetric solutions to ones with smaller isotropy groups. eg. cluster states.

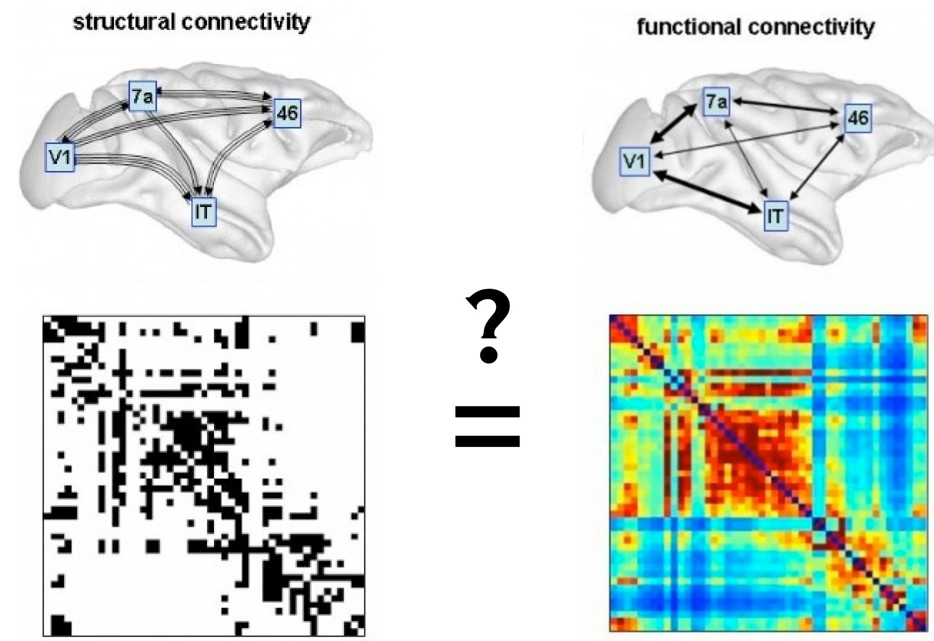
Synchrony $\lambda = -\epsilon H'(0)$ **Asynchrony** $\lambda_n = -2\pi i n \epsilon H_{-n}$



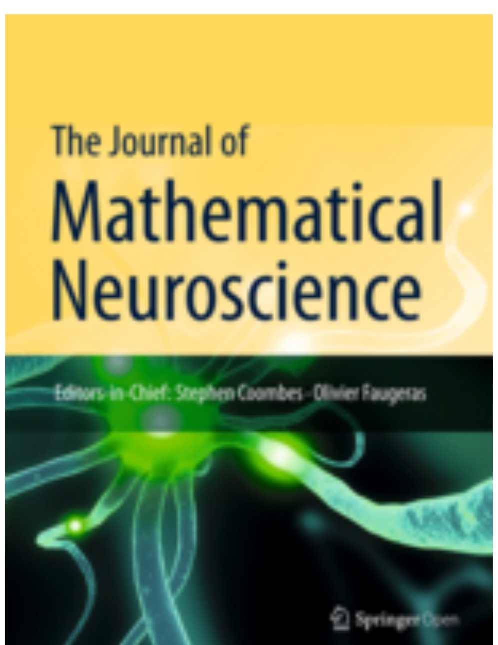
Phase oscillator networks in neuroscience

Biorobotics lab at EPFL
http://biorob.epfl.ch/

$$\dot{\theta}_i = \omega_i \quad + $$



$$\widehat{\mathcal{H}}_{ij}(\Phi) = \epsilon \left[w_{ij} H'(\phi_j - \phi_i) - \delta_{ij} \sum_k w_{ik} H'(\phi_k - \phi_i) \right]$$



Review
Mathematical Frameworks for Oscillatory Network Dynamics in Neuroscience
Ashwin P, Coombes S and Nicks R
The Journal of Mathematical Neuroscience 2016, **6**:2 (6 January 2016)